An integer programming approach to the kidney exchange problem

Carolina Riascos-Alvarez[†], Dionne M. Aleman[†]

[†]Department of Mechanical and Industrial Engineering, University of Toronto

INTRODUCTION

Chronic kidney disease (CKD) is a significant public health issue. In the 2015 Global Burden of Disease Study, overall CKD mortality has increased by 31.7% over the last 10 years, making it one of the fastest rising major causes of death [4]. As a result, the number of people in need of a kidney transplant is growing rapidly, and being on the wait list is one of the few possibilities to receive a transplant. Unfortunately, there are not enough deceased kidney donations to help everyone in need [1, 2].

1,600 — **In Canada alone, over 1,600 people are added to organ transplant wait lists every year.**

76% — **76% of the Canadian donation wait list are in need of a kidney transplant.**

44.3% — **Roughly 44.3% of patients on dialysis treatments survive at least five years.**

$ 1.9 billion — **Canada spends about $1.9 billion dollars yearly supporting patients on dialysis.**

Kidney paired donation (KPD) programs aim to increase living donor kidney transplantation as an alternative to the traditional deceased donor transplantation.

KIDNEY PAIRED DONATION (KPD)

When a living kidney donor is incompatible with the recipient, they are considered an incompatible patient-donor pair. A **paired kidney exchange** occurs when recipients swap their donors in order to get a compatible kidney. Potential donors and recipients can be **incompatible** due to **blood-type mismatch** and the presence of **antibodies** threatening the new organ tissue. Exchanges can have either the form of **cycles** or **chains** (Figure 1). Swaps only involving incompatible pairs have the form of cycles, whereas those involving both incompatible pairs and **altruistic donors** (NDD) have the form of chains.

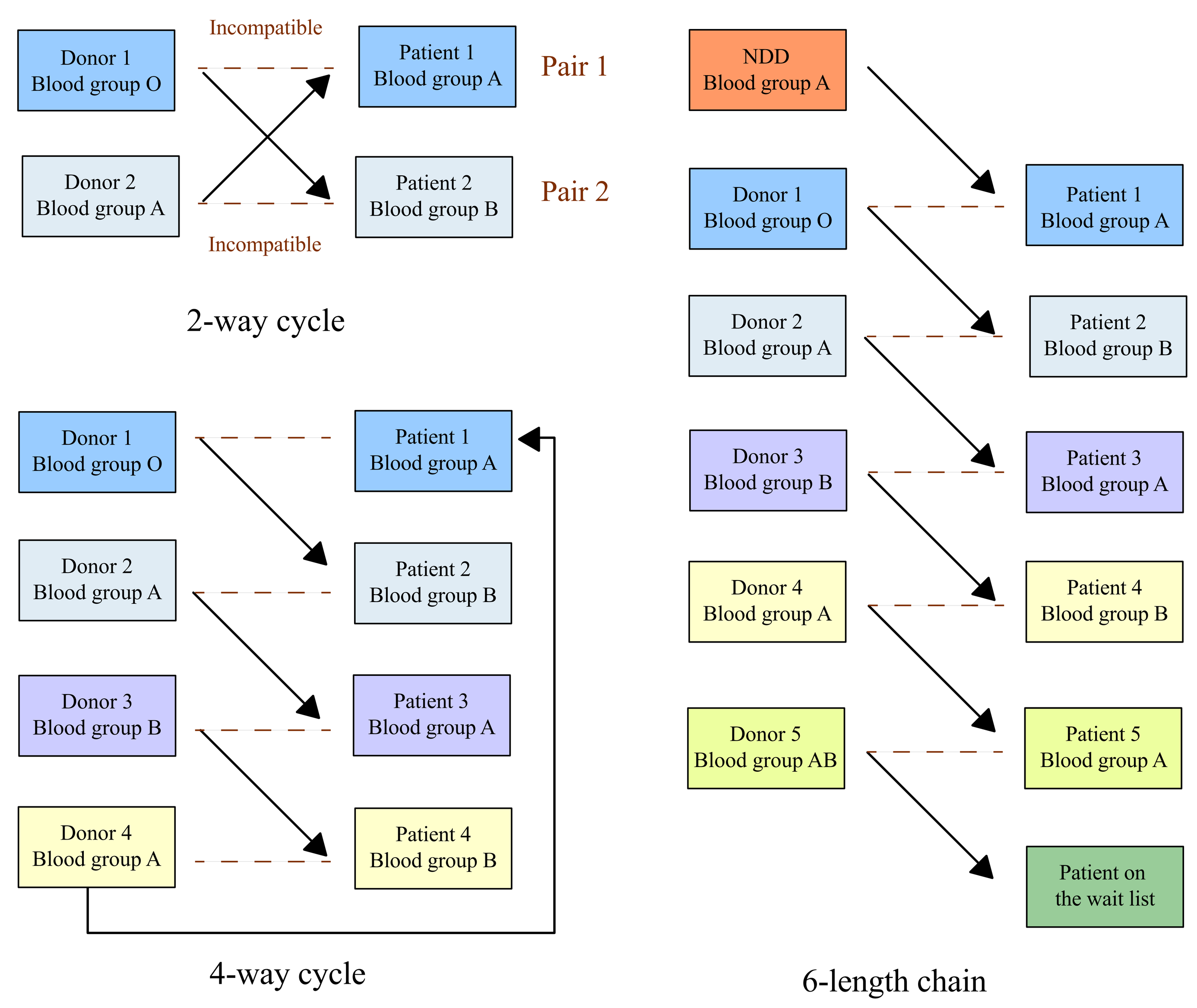

Figure 1: Types of exchanges

In practice, swaps in the form of cycles have to be performed simultaneously, resulting in complex logistic and capacity challenges. Therefore, the **size of a cycle**, k, is greater than three but usually less than or equal to six. **Chains may or may not be constrained**.

THE KIDNEY EXCHANGE PROBLEM

Incompatible pairs can also be represented as **nodes** and **compatible exchanges** as **directed edges** (arrows), whose end point indicates the direction in which a transplant is compatible.

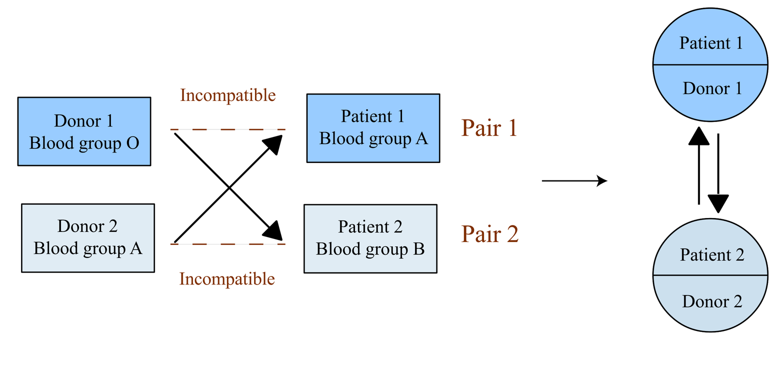

Figure 2: Pair-on-node representation

The pool of incompatible patient-donor pairs, altruistic donors, and the potential transplants between pairs, give rise to the **compatibility network** (Figure 3). Here, edges exist between compatible donors and recipients, and solid lines represent a feasible collection of kidney swaps. If we consider $k = 4$, then one cycle of size four (1-2-3-7) and one chain of size three (9-5-6-4) are obtained.

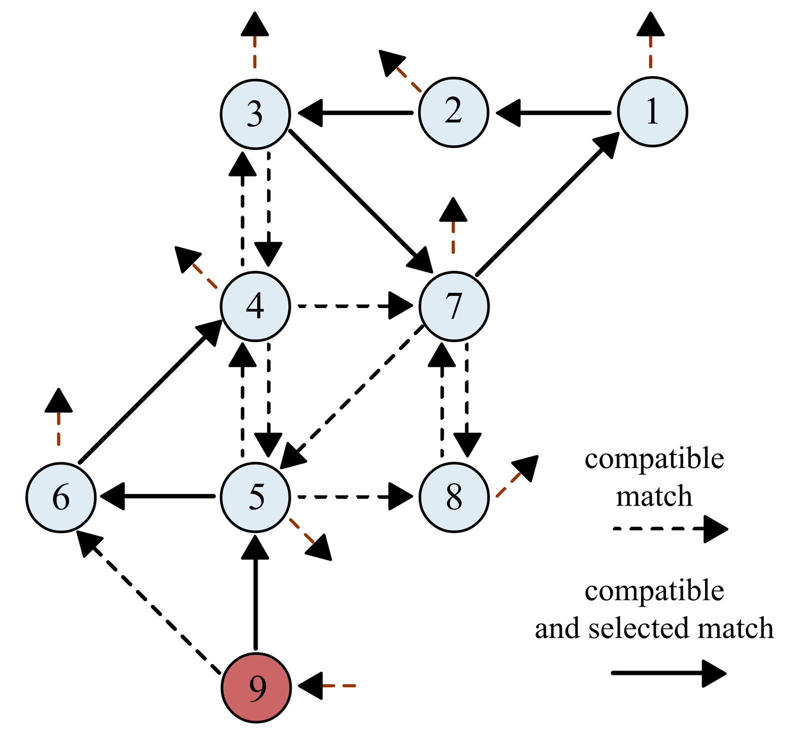

Figure 3: Compatible network, $w_{ij} = 1$

Definition **The kidney exchange problem is to find a maximum-weight node-disjoint union of directed cycles and chains in a compatibility network**.
The union must be node-disjoint so that no kidney is donated more than once, and maximum weight so that recipients with higher priority have a better chance to be transplanted.

MATHEMATICAL REPRESENTATION

Let $G = (V, E, w)$ be a compatibility network representing donors and recipients in a KPD program:

- V is the set of incompatible pairs and altruistic donors.
- E is the set of edges, such that there exists an edge between pair i and j if and only if the donor in pair i is compatible with the recipient in pair j.
- w_{ij} is a weight (priority) given to a transplant between pairs i and j.

Solving KEP on the full graph is extremely difficult, so we instead obtain solutions for sequential subgraphs $G_\ell = (V_\ell, E_\ell, w)$, with the final solution being optimal to the full graph [3]. This approach gives rise to the extended edge (EE) integer program [3], which we improve through the addition of bounds on the number of cycles in the final solution ($L = 1, ..., \ell, ...|V^*|$). We call our formulation the bounded extended edge (BEE) model, which is given by

$$x_{ij}^{\ell} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is used in the } \ell\text{-th copy of the graph} \\ 0 & \text{otherwise} \end{cases}$$

The bounded and extended edge formulation (BEE) of KEP is as follows:

$$\text{Maximize} \quad \sum_{\ell \in L} \sum_{(i,j) \in E_\ell} w_{ij} x_{ij}^{\ell} \qquad (1)$$

$$\text{subject to} \quad \sum_{j:(i,j)\in E_\ell} x_{ij}^{\ell} = \sum_{j:(j,i)\in E_\ell} x_{ji}^{\ell} \qquad i \in V, \ell \in L \qquad (2)$$

$$\sum_{\ell \in L} \sum_{j:(i,j)\in E_\ell} x_{ij}^{\ell} \le 1 \qquad i \in V \qquad (3)$$

$$\sum_{(i,j)\in E_\ell} x_{ij}^{l} \le k \qquad \ell \in L \qquad (4)$$

$$x_{ij}^{\ell} \in \{0, 1\} \qquad (i,j) \in E_\ell,\ \ell \in L \qquad (5)$$

RESULTS

We tested our formulation for several values of k and compared it with the existing extended edge formulation (EE) [3]. We generated 92 instances based on real historical data, whose characteristics are displayed in Table 1.

As observed in Figure 4, our model (BEE) is very consistent and effective to find the optimal solution in much less time than the EE formulation and always far below the time limit.

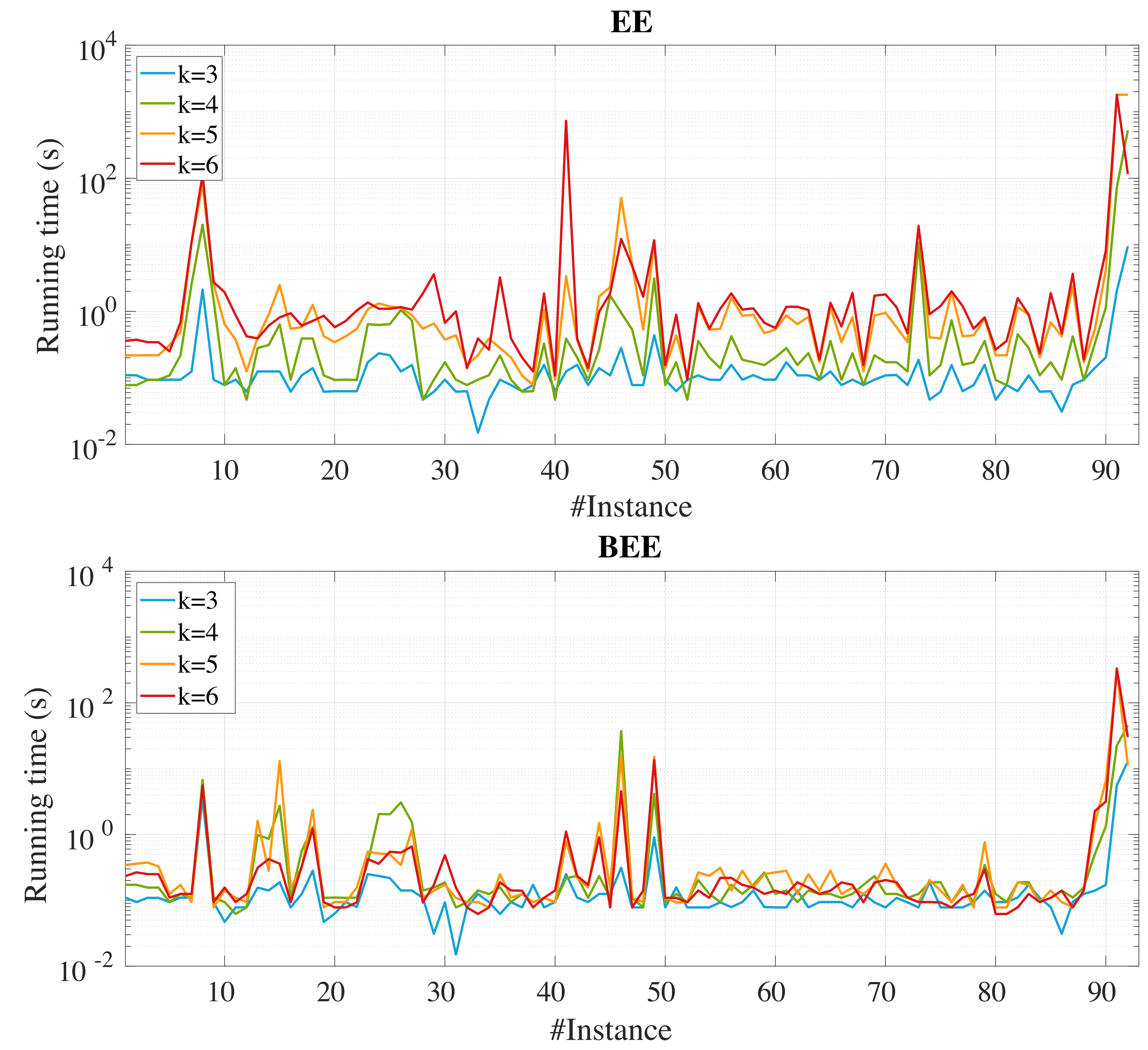

Figure 4: Computational performance

Table 1: Characteristics of the tested instances

	Min	Max
Pairs	114	474
Edges	1 519	34 772
Density	10%	18%

CONCLUSION

- Kidney paired donation is a well-established alternative to increase living donor kidney transplantation.
- Although the KEP is a hard-to-solve problem, advanced mathematical techniques can be applied to obtain optimal solutions quickly.
- Our approach was effective for the real-sized instances tested, outperforming another well-known integer programming formulation.

REFERENCES

[1] Canadian Institute for Health Information. High Risk and High Cost: Focus on Opportunities to Reduce Hospitalizations of Dialysis Patients in Canada. Technical report, 2016.

[2] Canadian Organ Replacement Register. CIHI Snapshot Annual Statistics on Organ Replacement in Canada. Technical report, 2017.

[3] Miguel Constantino, Xenia Klimentova, Ana Viana, and Abdur Rais. New insights on integer-programming models for the kidney exchange problem. *European Journal of Operational Research*, 231(1):57–68, nov 2013.

[4] GBD 2015 Mortality and Causes of Death Collaborators. Global, regional, and national life expectancy, all-cause mortality, and cause-specific mortality for 249 causes of death, 1980-2015: a systematic analysis for the Global Burden of Disease Study 2015. *Lancet (London, England)*, 388(10053):1459–1544, oct 2016.